Solution of Eq. (14) is

$$v_0 = \frac{2}{(\pi)^{1/2}} U(x) \left\{ \int_o^t e^t \left[\int_o^\eta e^{-\eta^2} d\eta \right] dt \right\} e^{-t} \tag{15}$$

which satisfies the initial condition $v_o = 0$ for $t = 0$. It is seen that, as $\eta \to \infty$, we obtain

$$v_o = U(x)[1 - e^{-t}] \tag{16}$$

This shows how the velocity of the dust particles differs with the potential velocity $U(x)$ of the fluid in the mainstream. With the passage of time this difference diminishes and as the motion stabilizes, $v_o \to U(x)$ as $t \to \infty$.

Taking the second approximation, Eqs. (7) and (8) become

$$\partial u_1/\partial t = (\partial^2 u_1/\partial y^2) + (v_o - u_o) \tag{17}$$

and

$$(\partial v_1/\partial t) + v_1 = u_1 \tag{18}$$

By similarity transformation $\eta = y/2(t)^{1/2}$ Eq. (17) becomes

$$(d^2 u_1/d\eta^2) + 2\eta(du_1/d\eta) = F(x, \eta, t) \tag{19}$$

where

$$F(x, \eta, t) = 4t(u_o - v_o)$$
$$= \frac{8}{(\pi)^{1/2}} U(x) t e^{-t} \int_o^\eta e^{-\eta^2} d\eta$$

Now solution of Eq. (19) is

$$u_1 = \int_o^\eta e^{-\eta^2} \left\{ \int e^{\eta^2} F(x, \eta, t) d\eta \right\} d\eta + C_1 \int_o^\eta e^{-\eta^2} d\eta + C_2$$

when

$$\eta = 0, \; u_1 = 0 \quad \therefore C_2 = 0$$

Since at $\eta = \infty$, $u_1 = 0$, then C_1 is given by the relation

$$C_1 = -\frac{16}{\pi} U(x) t e^{-t} \int_o^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left(\int_o^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \tag{20}$$

$$\therefore u_1 = \int_o^\eta e^{-\eta^2} \left\{ \int e^{\eta^2} F(x, \eta, t) d\eta \right\} d\eta + C_1 \int_o^\eta e^{-\eta^2} d\eta \tag{21}$$

where C_1 is given by relation (20).

Solution of Eq. (18) is given by

$$v_1 = e^{-t} \int_o^t u_1 e^t dt \tag{22}$$

Modifications of the velocity profiles of u and v are given by Eqs. (21) and (22), respectively.

Local Skin Friction

If $\tau_D(x)$ is the local skin friction for a dusty gas, then

$$\tau_D(x) = \mu(\partial u/\partial y)_{y=0}$$
$$= \frac{\mu U(x)}{(\pi t)^{1/2}} \left[1 - \frac{8 f t e^{-t}}{(\pi)^{1/2}} \int_o^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left(\int_o^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \right] \tag{23}$$

When f is small, the effect of dust particles is to increase the effective density of the gas. The viscosity remains nearly unaffected, though the kinematic viscosity is reduced by the factor $1/(1+f)$. These increases in density and the reduction of kinematic viscosity results are credited to P. G. Saffman.[2] If $\tau_o(x)$ be the local skin friction for the corresponding clean gas then

$$\tau_o(x) = \mu U(x)/(\pi t)^{1/2} \tag{24}$$

$$\therefore \frac{\tau_D(x)}{\tau_o(x)} = \left[1 - \frac{8f}{(\pi)^{1/2}} t e^{-t} \int_0^\infty e^{-\eta^2} \left\{ \int e^{\eta^2} \left(\int_0^\eta e^{-\eta^2} d\eta \right) d\eta \right\} d\eta \right] \tag{25}$$

From Eq. (25) it is seen that if f is nonzero the ratio $[\tau_D(x)/\tau_o(x)] < 1$. Thus, the dust particles reduce the local skin friction. This reduction diminishes as f diminishes and as t increases $\tau_D(x) \to \tau_o(x)$. However, all these analyses are valid (as in Rayleigh's case) in the initial development of the motion.

References

[1] Sproull, W. T., "Viscosity of Dusty Gases," *Nature*, Vol. 190, 1961, pp. 190, 976.

[2] Saffman, P. G., "On the Stability of Laminar Flow of a Dusty Gas," *Journal of Fluid Mechanics*, Vol. 13, Pt. 1, 1962, pp. 120–128.

[3] Michael, D. H., "Plane Parallel Flow of a Dusty Gas," *Mathematika*, Vol. 13, Pt. 1, 1966, pp. 97–109.

[4] Michael, D. H., "The Steady Motion of a Sphere in a Dusty Gas," *Journal of Fluid Mechanics*, Vol. 31, Pt. 1, 1968, pp. 175–192.

Influence of Viscosity on the Stability of a Cylindrical Jet

J. N. Anno*
University of Cincinnati, Cincinnati, Ohio

IN a previous Note,[1] Rayleigh's famous result[2] for the conditions of instability of an inviscid cylindrical jet was derived by considering an integral form of the conservation of mechanical energy. Given in this present Note is an extension of the integral method to an analysis of the conditions for instability of a viscous cylindrical jet. The result obtained is identical to that obtained by Weber[3] in 1931, but the derivation is believed to be more general. Specifically, it is not necessary to assume a jet at rest, nor to explicitly invoke the Navier-Stokes equations.

For a viscous fluid, an energy balance for any closed, stationary control volume of volume V and surface area S can be written as

$$\int_S \tau_{ij} n_j u_i \, dS = \int_S \frac{1}{2} \rho u_j n_j u_i u_i \, dS + \int_V \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u_i u_i \right) dV + \int_V R \, dV \tag{1}$$

where R is the viscous energy dissipation, given by

$$R = (\mu/2)(u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) \tag{2}$$

In this notation τ_{ij} is the stress tensor, u_i is the velocity component in the ith direction, n_j is the projection of the outward unit normal to the surface in the jth direction, ρ is the fluid density, μ is the fluid viscosity and, as conventional in tensor notation, repeated indices indicate summation over all coordinates. A comma within a subscript denotes differentiation with respect to the coordinate following it.

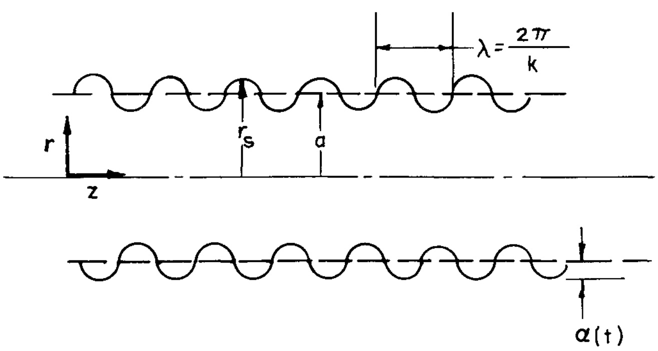

Fig. 1 Geometry of the cylindrical jet.

Received September 27, 1973; revision received March 18, 1974.
Index category: Hydrodynamics.

* Professor of Nuclear Engineering, Department of Chemical and Nuclear Engineering.

Consider a cylindrical jet as sketched in Fig. 1. For purposes of analysis, as both Rayleigh and Weber did, we consider a particular form of perturbation from the equilibrium cylindrical jet given by

$$r_s = a + \alpha(t)\cos kz \tag{3}$$

where a is the radius of the unperturbed cylinder, $\alpha(t)$ is a small time-dependent disturbance amplitude, and $k = 2\pi/\lambda$ is the wavenumber, λ being the wavelength of the perturbation. Cylindrical polar coordinates r, θ, z are being used. For simplicity, we assume no angular disturbance (θ-dependency).

For the case $R = 0$, the flow is irrotational and, as derived in the previous note,[1] the velocity components are

$$\begin{aligned} u_r &= \dot{\alpha}[I_1(kr)/I_1(ka)]\cos kz \\ u_z &= w_0 - \dot{\alpha}[I_0(kr)/I_1(ka)]\sin kz \end{aligned} \tag{4}$$

where w_0 is the velocity of the undisturbed jet and the dot denotes a time derivative. For this case, evaluation of Eq. (1) led to the result

$$\ddot{\alpha} = \alpha\left\{\frac{\gamma}{\rho a^3}\eta(1-\eta^2)\frac{I_1(\eta)}{I_0(\eta)}\right\} \tag{5}$$

where $\eta = ka$ and γ is the surface tension of the fluid. For $\eta < 1$, we have exponential solutions of the form $\alpha = \alpha_0 e^{\omega t}$, where

$$\omega^2 = (\gamma/2\rho a^3)(1-\eta^2)\eta^2 \tag{6}$$

and the approximation $I_1(\eta)/I_0(\eta) = \eta/2$ has been used.

By inspection of the $\mu = 0$ case [Eq. (4)], it is seen that u_z is only weakly dependent on r. Therefore, as an approximation, assume u_z is a function of z only. Then from the continuity equation

$$\partial u_z/\partial z + (1/r)(\partial/\partial r)(ru_r) = 0 \tag{7}$$

with the conditions that u_r be finite along the axis and that

$$(u_r)_{r=a} = \dot{\alpha}\cos kz \tag{8}$$

we have

$$\begin{aligned} u_r &= (r/a)\dot{\alpha}\cos kz \\ u_z &= w_0 - (2\dot{\alpha}/ka)\sin kz \end{aligned} \tag{9}$$

Now for both the inviscid and viscous free cylindrical jet

$$\tau_{rr} = -\left(\frac{\gamma}{r_s} + \frac{\gamma}{R}\right) = -\frac{\gamma}{a} + \frac{\alpha\gamma}{a^2}(1-k^2a^2)\cos kz \tag{10}$$

where $1/R$ is the curvature in the osculating plane. Further, for the freejet for $i \neq j$, $\tau_{ij} = 0$ at $r = a$, i.e. there are no surface shears. Selecting the stationary control volume over which to perform the integrations as the unperturbed jet; i.e., a cylinder with radius a, we can then evaluate the terms in the integral energy equation to be

$$\int_S \tau_{ij}n_j u_i\, dS = \frac{2n\pi^2\gamma\alpha\dot{\alpha}}{ka}(1-k^2a^2) \tag{11}$$

$$\int_S \frac{1}{2}\rho u_j n_j u_i u_i\, dS = 0 \tag{12}$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right)dV = \frac{n\pi^2\rho\dot{\alpha}\ddot{\alpha}}{2k^3}(8+k^2a^2) \tag{13}$$

$$\int_V R\, dV = \int_V 2\mu\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)^2\right]dV = \frac{n\pi^2\mu\dot{\alpha}^2}{2k}(24+k^2a^2) \tag{14}$$

The z integration has been taken over an even number of wavelengths, $z = 0$ to $2\pi n/k$, where n is an integer. Thus from Eqs. (11–14), the integral energy equation leads, after cancellation, to the differential equation

$$\ddot{\alpha} + \frac{\mu k^2}{\rho}\left(\frac{24+\eta^2}{8+\eta^2}\right)\dot{\alpha} - \frac{4\gamma\eta^2}{\rho a^3}\frac{(1-\eta^2)}{(8+\eta^2)}\alpha = 0 \tag{15}$$

For $\eta < 1$, the jet is unstable and for exponentially growing solutions $e^{\omega t}$, we have from Eq. (15)

$$\omega^2 + \frac{3\mu k^2\omega}{\rho} - \frac{\gamma}{2\rho a^3}(1-\eta^2)\eta^2 = 0 \tag{16}$$

Thus the effect of viscosity is to dampen the instability. However, only in the limit of infinite viscosity is the instability removed. It is further to be noted that since R is positive definite, a damping will always result, regardless of the exact choice of the velocity components. The damping coefficient $3\mu k^2/\rho$ is exactly the result obtained by Weber.[3] Note that for $\mu = 0$, Eq. (16) reduces to the Rayleigh result, Eq. (6), in the approximation $I_1(\eta)/I_0(\eta) = \eta/2$. For a very viscous jet, such that $(3\mu k^2/2\rho)^2 \gg (\gamma/2\rho a^3)$, the solution is

$$\omega = (\gamma/6\mu a)(1-\eta^2) \tag{17}$$

Longer wavelengths are thus more unstable for the viscous jet.

References

[1] Anno, J. N. and Walowit, J. A., "Integral Form of the Derivation of Rayleigh's Criterion for the Instability of an Inviscid Cylindrical Jet," *American Journal of Physics*, Vol. 38, No. 10, Oct. 1970, pp. 1255–1256.

[2] Lord Rayleigh, *The Theory of Sound*, 1st ed., Vol. 2, Dover, New York, 1945, pp. 351–359.

[3] Weber, C., "Zum Zerfall eines Flüssigkeitsstrahles," (Disintegration of a Liquid Jet), *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 11, No. 2, April 1931, pp. 136–153.

Penetration of Particles Injected into a Constant Cross Flow

George Rudinger*
Bell Aerospace Division of Textron, Buffalo, N.Y.

Nomenclature

C_D = drag coefficient
D = particle diameter
$F(\alpha)$ = function given by Eq. (11)
Re = Reynolds number
t = time
u = velocity in the direction of the gas flow
v = velocity across the gas flow
y = lateral distance from injection point
Y = y_{max} = particle penetration
α = parameter defined by Eq. (8)
μ = dynamic viscosity
ρ = density

Subscripts

o = at injection point
g = gas
p = particle
st = based on Stokes drag

Introduction

In the analysis of gas-particle flow for combustion chambers and other applications, it often becomes necessary to compute the penetration of particles injected into a cross flow. For low Reynolds numbers (less than about one), Stokes drag ($C_D = 24/Re$) can be applied. The relationship between the drag force

Received November 9, 1973. This paper is based on work sponsored in part by the Air Force Office of Scientific Research (AFSC), U.S. Air Force, under Contract F44620-70-C-0116 with technical monitoring by B. T. Wolfson. The U.S. Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Multiphase Flows; Nozzle and Channel Flow.

* Principal Scientist. Associate Fellow AIAA.